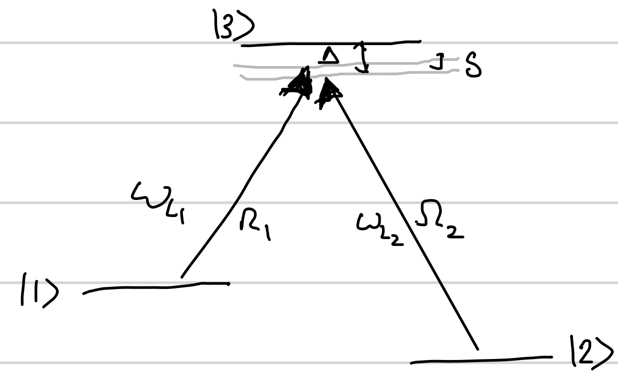
Physics 566 - Quantum Optics I
Problem Set 6 - Solutions

Problem 1: Λ-transitions and the Master Equation



$$\Delta = \Delta_1 = \omega_{L_1} - \frac{E_3 - E_1}{\hbar}$$
$$\Delta_2 = \omega_{L_2} - \frac{E_3 - E_2}{\hbar}$$
$$\delta = \Delta_1 - \Delta_2 = \omega_{L_1} - \omega_{L_2} - \frac{E_2 - E_1}{\hbar}$$

(a) The Hamiltonian $\hat{H} = \hat{H}_A - \hat{H}_{AL}$ in the Schrödinger Picture:

$$\hat{H}_A = E_1|1\rangle\langle 1| + E_2|2\rangle\langle 2| + E_3|3\rangle\langle 3|, \qquad \hat{H}_{AL} = \frac{\hbar\Omega_1}{2}\left(|3\rangle\langle 1|e^{-i\omega_{L_1}t} + |1\rangle\langle 3|e^{+i\omega_{L_1}t}\right) + \frac{\hbar\Omega_2}{2}\left(|3\rangle\langle 2|e^{-i\omega_{L_2}t} + |2\rangle\langle 3|e^{+i\omega_{L_2}t}\right)$$

We go to a "rotating frame" by making a unitary transformation $\hat{U} = \sum_{j=1}^{3} e^{-i\lambda_j t}|j\rangle\langle j|$

In the rotating frame: $\hat{H}_{RF} = \hat{U}^\dagger \hat{H}\hat{U} + \frac{\hbar}{-i}\frac{\partial U^\dagger}{\partial t}\hat{U} = \hat{U}^\dagger\hat{H}\hat{U} - \sum_j \hbar\lambda_j |j\rangle\langle j|$

$$\Rightarrow \hat{H}_{RF} = (E_1 - \hbar\lambda_1)|1\rangle\langle 1| + (E_2 - \hbar\lambda_2)|2\rangle\langle 2| + (E_3 - \hbar\lambda_3)|3\rangle\langle 3|$$
$$+ \frac{\hbar\Omega_1}{2}\left(|3\rangle\langle 1| e^{-i[\omega_{L_1} - (\lambda_3 - \lambda_1)]t} + h.c.\right) + \frac{\hbar\Omega_2}{2}\left(|3\rangle\langle 2| e^{-i(\omega_{L_2} - (\lambda_3 - \lambda_2))t} + h.c.\right)$$

We choose the rotating frame to make $\hat{H}_{RF}$ time independent

$$\Rightarrow \lambda_3 - \lambda_1 = \omega_{L_1}, \quad \lambda_3 - \lambda_2 = \omega_{L_2}$$

This is two equations for three unknowns. We can arbitrarily set the zero of energy.
$\Rightarrow$ Chose $\lambda_1 = E_1/\hbar \Rightarrow \lambda_3 = \omega_{L_1} + E_1/\hbar$, $\lambda_2 = -\omega_{L_2} + \lambda_3 = \omega_{L_1} - \omega_{L_2} + E_1/\hbar$

$$\boxed{\Rightarrow \hat{H}_{RF} = -\hbar\delta|2\rangle\langle 2| - \hbar\Delta|3\rangle\langle 3| + \frac{\hbar\Omega_1}{2}\left(|3\rangle\langle 1| + |1\rangle\langle 3|\right) + \frac{\hbar\Omega_2}{2}\left(|3\rangle\langle 2| + |2\rangle\langle 3|\right)}$$

(b) Given: Level $|3\rangle$ decays to $|1\rangle$ and $|2\rangle$ with rates Γ_{31} and Γ_{32} respectively. The total decay rate is $\Gamma = \Gamma_{31} + \Gamma_{32}$, so the effective (non-Hermitian) Hamiltonian that includes decay of $|3\rangle$ according to an imaginary part of the excited state eigenvalue: $\hat{H}_{eff} = \hat{H} - i\frac{\hbar\Gamma}{2}|3\rangle\langle 3|$. The Master equation that describes the evolution of the density operator:

$$\frac{d}{dt}\hat{\rho} = \frac{-i}{\hbar}\left(\hat{H}_{eff}\hat{\rho} - \hat{\rho}\hat{H}_{eff}^{\dagger}\right) + \mathcal{L}_{feed}[\hat{\rho}]$$

$$\mathcal{L}_{feed}[\hat{\rho}] = \underbrace{\Gamma_{31}\rho_{33}|1\rangle\langle 1|}_{\text{Feeding level } |1\rangle} + \underbrace{\Gamma_{32}\rho_{33}|2\rangle\langle 2|}_{\text{Feeding level } |2\rangle}$$

We can find the equations of motion for the density matrix most easily using the non-Hermitian Schrödinger equation: $\frac{\partial}{\partial t}|\psi\rangle = \frac{-i}{\hbar}\hat{H}_{eff}|\psi\rangle$, which including the decay, but not feeding.

$$|\psi\rangle = c_1|1\rangle + c_2|2\rangle + c_3|3\rangle$$

$$\Rightarrow \dot{c}_1 = \frac{-i}{\hbar}\langle 1|\hat{H}_{eff}|\psi\rangle = -i\frac{\Omega_1}{2}c_3; \quad \dot{c}_2 = -\frac{i}{\hbar}\langle 2|\hat{H}_{eff}|\psi\rangle = i\delta c_2 - \frac{i}{2}\Omega_2 c_3;$$

$$\dot{c}_3 = -\frac{i}{\hbar}\langle 3|\hat{H}_{eff}|\psi\rangle = \left(i\Delta - \frac{\Gamma}{2}\right)c_3 - \frac{i}{2}\Omega_1 c_1 - \frac{i}{2}\Omega_2 c_2.$$

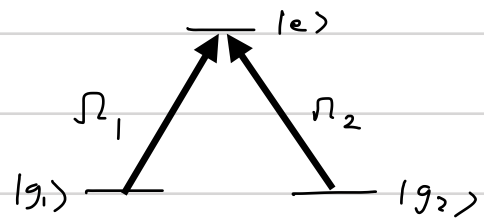
$$\rho_{\alpha\beta} = c_\alpha c_\beta^* \qquad \dot{\rho}_{\alpha\beta} = \dot{c}_\alpha c_\beta^* + c_\alpha \dot{c}_\beta^* + \langle\alpha|\mathcal{L}_{feed}[\hat{\rho}]|\beta\rangle$$

$$\boxed{\begin{aligned}
\frac{d}{dt}\rho_{11} &= \dot{c}_1 c_1^* + c_1\dot{c}_1^* + \frac{d}{dt}\rho_{11}\Big|_{feed} = -i\frac{\Omega_1}{2}(\rho_{31} - \rho_{13}) + \Gamma_{31}\rho_{33}\\
\frac{d}{dt}\rho_{22} &= \dot{c}_2 c_2^* + c_2\dot{c}_2^* + \frac{d}{dt}\rho_{22}\Big|_{feed} = -\frac{i}{2}\Omega_2(\rho_{32} - \rho_{23}) + \Gamma_{32}\rho_{33}\\
\frac{d}{dt}\rho_{33} &= \dot{c}_3 c_3^* + c_3\dot{c}_3^* = -\Gamma\rho_{33} - \frac{i}{2}\Omega_1(\rho_{13} - \rho_{31}) - \frac{i}{2}\Omega_2(\rho_{23} - \rho_{32})\\
\frac{d}{dt}\rho_{23} &= \dot{c}_2 c_3^* + c_2\dot{c}_3^* = i\left(\delta - \Delta + i\frac{\Gamma}{2}\right)\rho_{23} - \frac{i}{2}\Omega_2(\rho_{33} - \rho_{22}) + \frac{i}{2}\Omega_1\rho_{21}\\
\frac{d}{dt}\rho_{13} &= \dot{c}_1 c_3^* + c_1\dot{c}_3^* = \left(-i\Delta - \frac{\Gamma}{2}\right)\rho - \frac{i}{2}\Omega(\rho_{33} - \rho_{11}) + \frac{i}{2}\Omega_2\rho_{12}\\
\frac{d}{dt}\rho_{12} &= \dot{c}_1 c_2^* + c_1\dot{c}_2^* = -i\delta\rho_{12} - i\frac{\Omega_1}{2}\rho_{32} + \frac{i}{2}\Omega_2\rho_{13}
\end{aligned}}$$

These equations determine the full dynamics of the three-level system, including decay and refeeding of population by optical pumping. However, when the saturation parameter is low, often one can obtain the important physics solely from the non-Hermitian Schrödinger equation.

Problem 2: Dark States

We consider the 3-level "lambda system" with two ground states resonantly coupled to an excited state.



In the rotating frame: $\hat{H}_{RF} = \frac{\hbar\Omega_1}{2}\left(|g_1\rangle\langle e| + |e\rangle\langle g_1|\right) + \frac{\hbar\Omega_2}{2}\left(|g_2\rangle\langle e| + |e\rangle\langle g_2|\right)$

(a) The "dressed states" are the eigenstates of the coupled atom-laser system

The Hamiltonian matrix: $\hat{H}_{RF} = \frac{\hbar}{2}\begin{bmatrix} 0 & 0 & \Omega_1 \\ 0 & 0 & \Omega_2 \\ \Omega_1 & \Omega_2 & 0 \end{bmatrix}$ In the ordered basis $\{|g_1\rangle, |g_2\rangle, |e\rangle\}$

The eigenvectors / eigenvalues of this matrix are:

$$|\psi_{Dark}\rangle = \frac{-\Omega_2|1\rangle + \Omega_1|2\rangle}{\sqrt{\Omega_1^2+\Omega_2^2}}, \qquad E_{Dark} = 0 \qquad \text{(Dark state)}$$

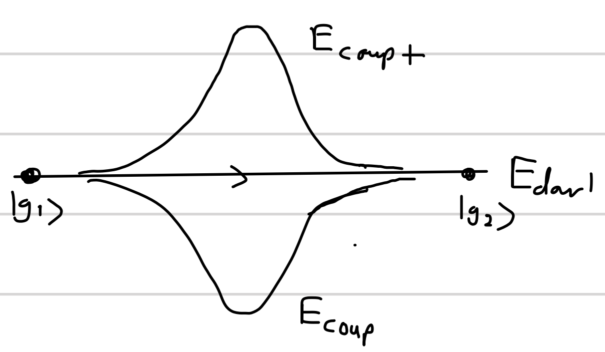
$$|\psi_{Coup,\pm}\rangle = \frac{1}{\sqrt{2}}\left(|\psi_{Bright}\rangle \pm |e\rangle\right), \qquad E_{coup,\pm} = \pm\frac{\hbar}{2}\sqrt{\Omega_1^2+\Omega_2^2} \qquad \text{(Coupled states)}$$

$(\Omega_1|1\rangle + \Omega_2|2\rangle)/\sqrt{\Omega_1^2+\Omega_2^2}$

(b) One can transfer population from $|g_1\rangle$ to $|g_2\rangle$ using the "counter-intuitive pulse sequence".

First apply field that couples $|2\rangle \to |3\rangle$.
As we turn that field off turn on $|1\rangle \to |3\rangle$

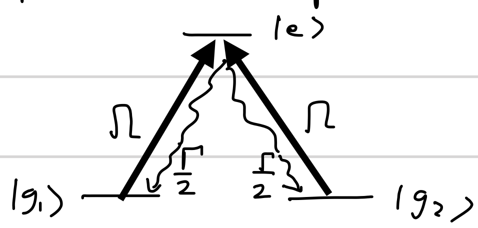
The counter-intuitive pulse sequence achieves the transfer from $|g_1\rangle \to |g_2\rangle$ by adiabatic following of the dark state. The dressed eigenvalues as a function of time appear as:



At $t=0$ $|\psi\rangle = |g_1\rangle$. As we slowly turn on Ω_2, $|\psi\rangle$ is in the dark state (the eigenstate of $\hat{H}$ with $\Omega_1 = 0$)). If we turn on and off the fields slowly, so the transition is adiabatic, then $|\psi(t)\rangle$ follows $|\psi_{Dark}(t)\rangle = \dfrac{\Omega_2(t)|2\rangle - \Omega_1(t)|1\rangle}{\sqrt{\Omega_1^2(t)+\Omega_2^2(t)}}$. Intermediately, when

$\Omega_1 = \Omega_2$ $|\psi_{Dark}\rangle = \dfrac{|1\rangle - |2\rangle}{\sqrt{2}}$. At the end of pulses $|\psi(t)\rangle = |\psi_{Dark}(t\to\infty)\rangle = |2\rangle$.

(c) We now consider spontaneous decay. We simplify to the case $\Omega_1 = \Omega_2 = \Omega$; $\Gamma_{31} = \Gamma_{32} = \frac{\Gamma}{2}$



In the basis $\{|D\rangle, |B\rangle, |e\rangle\}$ where $|D\rangle = \frac{|1\rangle - |2\rangle}{\sqrt{2}}$, $|B\rangle = \frac{|1\rangle + |2\rangle}{\sqrt{2}}$, $|e\rangle$

$|1\rangle = \frac{1}{\sqrt{2}}(|D\rangle + |B\rangle)$; $|2\rangle = \frac{1}{\sqrt{2}}(|B\rangle - |D\rangle)$

$$\Rightarrow \hat{H} = \frac{\sqrt{2}\hbar\Omega}{2}\left(|B\rangle\langle e| + |e\rangle\langle B|\right)$$

(effective 2-level coupling between $|B\rangle$ & $|e\rangle$ w/ Rabi frequency $\sqrt{2}\Omega$. The Dark State is decoupled)

the effective Hamiltonian $\hat{H}_{eff} = \hat{H} - i\frac{\hbar}{2}\Gamma\, |e\rangle\langle e|$

The "Feeding term" in the Master Equation: $\mathcal{L}_{feed}[\hat{\rho}] = \frac{\Gamma}{2}\rho_{ee}\left(|1\rangle\langle 1| + |2\rangle\langle 2|\right) = \frac{\Gamma}{2}\rho_{ee}\left(|B\rangle\langle B| + |D\rangle\langle D|\right)$

(d) The Master equation: $\frac{d\hat{\rho}}{dt} = \frac{-i}{\hbar}\left(\hat{H}_{eff}\hat{\rho} - \hat{\rho}\hat{H}_{eff}^{+}\right) + \mathcal{L}_{feed}[\hat{\rho}]$

Non-Hermitian Schrödinger Eqn: $\frac{\partial}{\partial t}|\psi\rangle = \frac{-i}{\hbar}\hat{H}_{eff}|\psi\rangle$

$\Rightarrow \dot{c}_e = -\frac{\Gamma}{2}c_e + i\frac{\sqrt{2}\Omega}{2}c_B$, $\dot{c}_B = i\frac{\sqrt{2}\Omega}{2}c_e$, $\dot{c}_D = 0$

$$\boxed{\begin{aligned}
\Rightarrow \dot{\rho}_{ee} &= \dot{c}_e c_e^* + c_e\dot{c}_e^* = -\Gamma\rho_{ee} + i\sqrt{2}\frac{\Omega}{2}\left(\rho_{Be} - \rho_{eB}\right)\\
\dot{\rho}_{BB} &= \dot{c}_B c_B^* + \dot{c}_B^* c_B + \langle B|\mathcal{L}_{feed}[\hat{\rho}]|B\rangle = +\frac{\Gamma}{2}\rho_{ee} + i\frac{\sqrt{2}\Omega}{2}\left(\rho_{eB} - \rho_{Be}\right)\\
\dot{\rho}_{DD} &= \dot{c}_D c_D^* + c_D\dot{c}_D^* + \langle D|\mathcal{L}_{feed}[\hat{\rho}]|D\rangle = \frac{\Gamma}{2}\rho_{DD}\\
\dot{\rho}_{eB} &= \dot{c}_e c_B^* + c_e\dot{c}_B^* = -\frac{\Gamma}{2}\rho_{eB} + i\frac{\sqrt{2}\Omega}{2}\left(\rho_{ee} - \rho_{BB}\right)\\
\dot{\rho}_{eD} &= \dot{c}_e c_D^* + c_e\dot{c}_D^* = -\frac{\Gamma}{2}\rho_{eD} + i\frac{\sqrt{2}\Omega}{2}\rho_{BD}\\
\dot{\rho}_{BD} &= \dot{c}_B c_D^* + c_B\dot{c}_D^* = i\frac{\sqrt{2}\Omega}{2}\rho_{eD}
\end{aligned}}$$

The steady-state solution: $\dot{\rho}_{BB} = \frac{\Gamma}{2}\rho_{ee} = 0 \Rightarrow \rho_{ee} = 0$

$$\Rightarrow \dot{\rho}_{ee} = i\frac{\sqrt{2}\Omega}{2}(\rho_{eB} - \rho_{Be}) = 0 \Rightarrow \mathrm{Im}(\rho_{eB}) = 0$$

$\Rightarrow \dot{\rho}_{eB} = -\frac{\Gamma}{2}\rho_{eB} - i\frac{\sqrt{2}\Omega}{2}\rho_{BB} = 0 \implies \mathrm{Re}(\rho_{eB}) = 0$ and $\rho_{BB} = 0$ (since Real & Imag $= 0$)

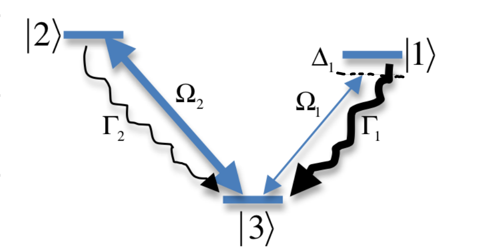
$$\left.\begin{aligned} \dot{\rho}_{eD} &= \dot{c}_e c_D^* + c_e \dot{c}_D^* = -\frac{\Gamma}{2}\rho_{eD} + i\frac{\sqrt{2}\Omega}{2}\rho_{BD} = 0 \\ \dot{\rho}_{BD} &= \dot{c}_B c_D^* + c_B \dot{c}_D^* = i\frac{\sqrt{2}\Omega}{2}\rho_{eD} = 0 \end{aligned}\right\} \Rightarrow \rho_{eD} = 0,\ \rho_{BD} = 0$$

$\Rightarrow$ Dynamically ALL elements of $\hat{\rho} \to 0$ in steady state except ρ_{DD}

$\Rightarrow$ Since the equation is trace preserving, in steady state, $\hat{\rho}_{s.s.} = |D\rangle\langle D|$

Problem 3: Autler–Townes Splitting

We consider a V-configuration:



A strong field on the $|2\rangle \leftrightarrow |3\rangle$ transition "dresses" these states. We then probe these dressed states with a weak probe on the $|3\rangle \leftrightarrow |1\rangle$ transition and measure the absorption as a function of detuning, Δ_1.

(a) The (trace-preserving) master equation for this 3-level atom is:

$$\frac{d}{dt}\hat{\rho} = -\frac{i}{\hbar}\left(\hat{H}_{eff}\hat{\rho} - \hat{\rho}\hat{H}^{\dagger}_{eff}\right) + \left(\Gamma_1\rho_{11} + \Gamma_2\rho_{22}\right)|3\rangle\langle 3|$$

where $\hat{H}_{eff} = -\hbar\left(\Delta_1 + i\frac{\Gamma_1}{2}\right)|1\rangle\langle 1| - \frac{i}{2}\hbar\Gamma_2\, |2\rangle\langle 2| + \frac{\hbar\Omega_1}{2}\left(|3\rangle\langle 1| + |1\rangle\langle 3|\right) + \frac{\hbar\Omega_2}{2}\left(|3\rangle\langle 2| + |2\rangle\langle 3|\right)$

Non-unitary evolution of probability amplitudes: $\frac{d}{dt}c_\alpha = -\frac{i}{\hbar}\langle\alpha|\hat{H}_{eff}|\psi\rangle$, $|\psi\rangle = \sum_{\alpha=1}^{3} c_\alpha|\alpha\rangle$

$$\Rightarrow \dot{c}_1 = \left(i\Delta_1 - \frac{\Gamma_1}{2}\right)c_1 - i\frac{\Omega_1}{2}c_3, \quad \dot{c}_2 = -\frac{\Gamma_2}{2}c_2 - i\frac{\Omega_2}{2}c_3, \quad \dot{c}_3 = -i\frac{\Omega_1}{2}c_1 - i\frac{\Omega_2}{2}c_2$$

The Master equation: $\dot{\rho}_{\alpha\beta} = \dot{c}_\alpha c^*_\beta + c_\alpha \dot{c}^*_\beta + (\Gamma_1\rho_{11} + \Gamma_2\rho_{22})\delta_{\alpha 3}\delta_{\beta 3}$

$$\Rightarrow \dot{\rho}_{11} = \dot{c}_1 c_1^* + c_1\dot{c}_1^* = -\Gamma_1\rho_{11} - i\frac{\Omega_1}{2}(\rho_{31} - \rho_{13}), \quad \dot{\rho}_{22} = \dot{c}_2 c_2^* + c_2\dot{c}_2^* = -\Gamma_2\rho_{22} - i\frac{\Omega_2}{2}(\rho_{32} - \rho_{23})$$

$$\dot{\rho}_{33} = -\dot{\rho}_{11} - \dot{\rho}_{22} = -(\Gamma_1\rho_{11} + \Gamma_2\rho_{22}) + i\frac{\Omega_1}{2}(\rho_{31} - \rho_{13}) + i\frac{\Omega_2}{2}(\rho_{32} - \rho_{23})$$

$$\dot{\rho}_{13} = \dot{c}_1 c_3^* + c_1\dot{c}_3^* = \left(i\Delta_1 - \frac{\Gamma_1}{2}\right)\rho_{11} - i\frac{\Omega_1}{2}(\rho_{33} - \rho_{11}) + i\frac{\Omega_2}{2}\rho_{12}$$

$$\dot{\rho}_{23} = \dot{c}_2 c_3^* + c_2\dot{c}_3^* = -\frac{\Gamma}{2}\rho_{23} - \frac{i}{2}\Omega_2(\rho_{33} - \rho_{22}) + i\frac{\Omega_1}{2}\rho_{21}$$

$$\dot{\rho}_{12} = \dot{c}_1 c_2^* + c_1\dot{c}_2^* = \left(i\Delta_1 - \frac{\Gamma_1 + \Gamma_2}{2}\right)\rho_{12} + \frac{i}{2}\Omega_2\rho_{13} - \frac{i}{2}\Omega_1\rho_{32}$$

(b) Under the assumption of weak excitation of $|1\rangle$, retain terms only linear in Ω_1.

The state state solutions, $\dot{\rho}_{\alpha\beta} = 0$

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$-\Gamma_1 \rho_{11} - \frac{i}{2}\Omega_1(\rho_{31}-\rho_{13}) = 0$ $\Rightarrow \rho_{11} = \frac{\Omega_1}{\Gamma_1}\left(\frac{\rho_{13}-\rho_{31}}{2i}\right) = \frac{\Omega_1}{\Gamma}\,\mathrm{Im}(\rho_{13})$

$(i\Delta_1 - \frac{\Gamma_1}{2})\rho_{13} - \frac{i\Omega_1}{2}(\rho_{33} - \underbrace{\cancel{\rho_{11}}}_{O(\Omega_1)}) + \frac{i}{2}\Omega_2\rho_{12}$ $\Rightarrow \rho_{13} \approx \frac{-\Omega_1}{(2\Delta_1+i\Gamma_1)}\rho_{33} + \frac{\Omega_2}{(2\Delta_1+i\Gamma_1)}\rho_{21}$

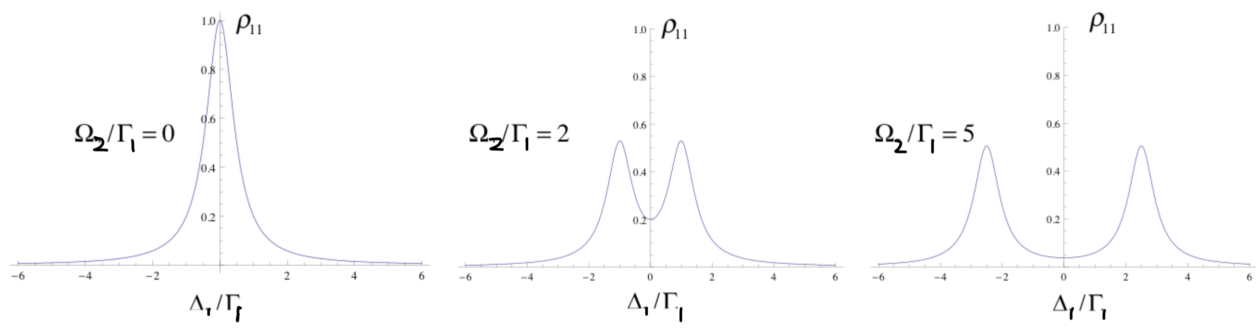
$(i\Delta_1 - \frac{\Gamma}{2})\rho_{12} + \frac{i}{2}\Omega_2\rho_{13} - \frac{i}{2}\underbrace{\cancel{\Omega_1\rho_{32}}}_{O(\Omega_1^2)}$ $\Rightarrow \rho_{12} \approx \left(\frac{\Omega_2}{2\Delta_1+i\Gamma_1}\right)\rho_{13}$

(c) Putting this all together:

$$\rho_{13} \approx \frac{-\Omega_1}{2\Delta_1+i\Gamma_1}\rho_{33} + \frac{\Omega_2^2}{(2\Delta_1+i\Gamma_1)^2}\rho_{13} \Rightarrow \rho_{13} \approx \frac{-\Omega_1(2\Delta_1+i\Gamma_1)}{(2\Delta_1+i\Gamma_1)^2-\Omega_2^2}\rho_{33}$$

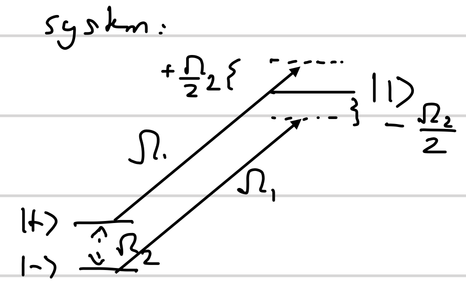
$$\Rightarrow \boxed{\rho_{11} = -\frac{\Omega_1^2}{\Gamma_1^2}\rho_{33}\,\mathrm{Im}\left[\frac{2\Delta_1+i\Gamma_1}{(2\Delta_1+i\Gamma_1)^2-\Omega_2^2}\right]}$$

Plots ρ_{11} as a function of Δ_1, normalized in units of $\frac{\Omega_1^2}{\Gamma_1^2}\rho_{33}$, for different coupling strengths on the auxiliary transition, Ω_2, are shown below.



(d) The Autler-Towns splitting is understood in the Dressed-basis of the strongly-coupled $|2\rangle - |3\rangle$ system:

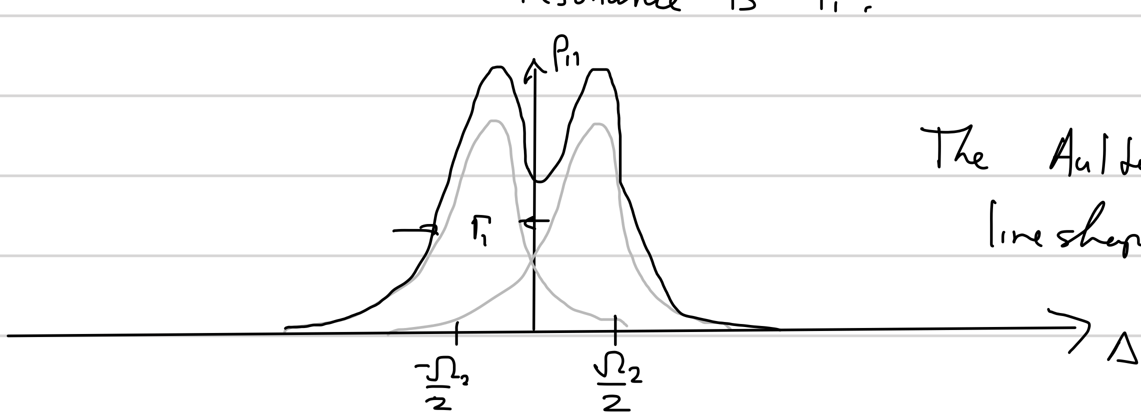
Sketch for $\Delta_1 = 0$



There are two resonances $|+\rangle \to |1\rangle$, $|-\rangle \to |2\rangle$

$|\pm\rangle = \frac{|2\rangle \pm |3\rangle}{\sqrt{2}}$

These two resonances are split by the dressed state splitting $\pm\frac{\Omega_2}{2}$. The width of each resonance is Γ_1.



The Autler Townes line shape

(c) The difference between EIT and Autler-Townes splitting is that the former is an interference effect arising from the destructive interference of two paths, whereas Autler-Townes is the incoherent sum of two distinguishable paths. In EIT the absorption goes exactly to zero at $\Delta_1 = 0$ whereas in Autler-Townes, it is reduced.

When $\Omega_2 \gg \Gamma_1$, the two paths of EIT are essentially distinguishable — EIT and Autler-Townes absorption spectra look essentially the same in this situation, i.e., two separated Lorentzians, with essentially zero absorption between.